Section 2.2

Focus will be to utilize explicit solutions from simple DEs to draw conclusions about equations that are difficult to solve.

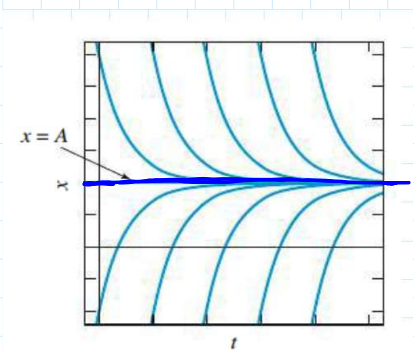
ex. Let $x(t)$ denote body temperature w init. temp. $x(0) = x_0$ at $t=0$. Body is immersed in a medium w constant temp, A.

Assume Newton's Law of Cooling:

$$\frac{dx}{dt} = -k(x-A)$$ where k is a positive constant

solution is $x(t) = A + (x_0 - A)e^{-kt}$ ← show yourself later

note that as $t \to \infty$ $x(t) \to A$



note: $x(t) \equiv A$ is also a sol'n

represents temp. of body when it's in thermal equilibrium with the medium that it's in / surrounded by

corresponding phase diagram:

scenario 1: $x < A$, $x' > 0$

scenario 2: $x > A$, $x' < 0$

both scenarios approach limiting solution: $x(t) \equiv A$ as t increases

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last week saw pop'n eqn: $\frac{dx}{dt} = (\beta - \delta)x$ where β, δ are birth, death rates respectively

δ vs Δ
lower upper

where β, δ are birth, death rates respectively

assuming β, δ are known functions of x ↑ lower case ↑ upper case

then eq'n CBW as, $\frac{dx}{dt} = f(x)$

considered an autonomous 1^{ST} order DE

:= independent variable t does not appear

see that solutions of $f(x)=0$ are significant

called critical points (CP) of autonomous DE $\frac{dx}{dt} = f(x)$

$f'(x)=0$

can conclude that if $x = c$ is CP

then DE has a constant sol'n $x(t) \equiv c$

sometimes considered an equilibrium solution

ex. pop'n stays constant because it's in "equilibrium" with its environment

ex. consider $\frac{dx}{dt} = kx(M-x)$ where $k>0$, $M>0$

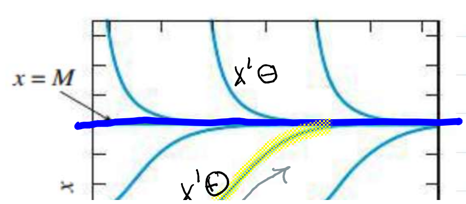
stand to reason that there are 2 CPs:

$$f(x) = \underbrace{kx}\,\underbrace{(M-x)} = 0$$

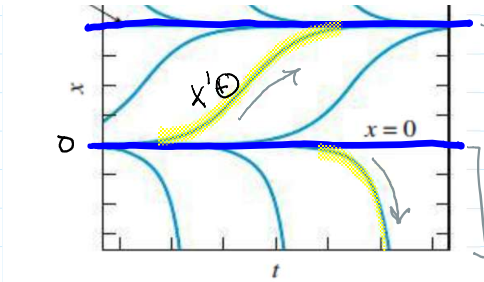
$$CP: \quad x=0 \quad x=M$$

"recall" $x(t) = \frac{Mx_0}{x_0 + (M-x_0)e^{-kMt}}$ satisfies $x(0) = x_0 \neq M$

now can also say that initial values $x_0=0$ and $x_0=M$ yield sol'ns



$x_0 > 0$ then $x(t) \to M$ as $t \to +\infty$



$x_0 > 0$ then $x(t) \to M$ as $t \to +\infty$

$x_0 < 0$ although denominator is initially positive, eventually it vanishes at some $t = t_1$, then as $t \to t_1$, $x(t) \to -\infty$

relevant <u>phase diagram</u>
for $\frac{dx}{dt} = kx(M-x)$

$x=0$ $\qquad$ $x=M$

$x'<0$ $\qquad$ $x'>0$ $\qquad$ $x'<0$

label as stable at this CP $x=M$

label as unstable at CP $x=0$

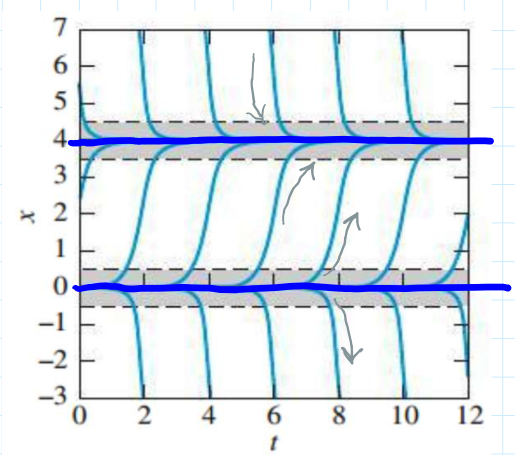
<u>Stability at CPs</u>

recall $\frac{dx}{dt} = 4x - x^2$ $\qquad$ similar to logistic equation

$$\frac{dP}{dt} = kP(M-P)$$
$$= kMP - kP^2$$

the solution curve looks like:

where $k=1$, $M=4$



CP $x=4$ $\quad$ See how solution curves "funnel" towards $x=4$

CP $x=0$ $\quad$ but curves diverge (act as a "spout") near $x=0$

in "funnel" situation, $x=c$ is considered stable
in "spout" " $x=c$ " unstable

Section 2.3 Acceleration - Velocity Model

ex. suppose a crossbow bolt is shot straight upward from ground $(y_0 = 0)$
given $V_0 = 49$ m/s

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then $\frac{dv}{dt} = -9.8 \longrightarrow v(t) = -9.8t + V_0$

a

$$\boxed{v(t) = -9.8t + 49}$$

$\therefore$ height $y(t)$ would be: $\int(-9.8t + 49)\,dt$

$$\boxed{y(t) = -4.9t^2 + 49t} + \cancel{y_0}^{\,0}$$

bolt reaches its max. height when $v = 0$: $-9.8t + 49 = 0$

$49 = 9.8t$

$\boxed{t = 5\text{ s}}$

can also say y_{max} is $y(5) = -4.9(25) + 49(5)$

$= -49(2.5) + 49(5)$

$= 49(5 - 2.5)$

$= 49(2.5)$

$= \boxed{122.5 \text{ m}}$

finally, bolt returns to ground when $y = 0$: $-4.9t^2 + 49t = 0$

$\cancel{t}(-4.9t + 49) = 0$

irrelevant sol'n

$49 = 4.9t$

$\frac{490}{49} = t$

$\boxed{t = 10\text{ s}}$

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to solve previous example used $F = ma$

$F \to F_G$ force of gravity $\longrightarrow$ equals $-mg$ where $g = 9.8\text{ m/s}^2$

$a \to \frac{dv}{dt}$

assumed no air resistance

now consider $m\frac{dv}{dt} = F_G + F_R$ simplistically, now consider $m\frac{dv}{dt} = F_G + F_R$

simplistically, F_R proportional to square of v: $F_R = kv^2$

realistically, $F_R = kv^p$ where $1 \le p \le 2$

k is dep. on size/shape of body as well as density/viscosity of air

let $p = 1$ then $F_R = -kv$ where k is a positive constant

$\frac{dy}{dt}$

if $v<0$ then $F_R > 0$ ← upward force
(body is falling)
if $v>0$ then $F_R < 0$ ← downward force
(body is rising)

so net force acting on body is: $F = F_R + F_G$

$F = -kv - mg$

then $m\frac{dv}{dt} = -kv - mg$ divide by m

$\frac{dv}{dt} = -\frac{k}{m}v - g$ let $\rho = \frac{k}{m} > 0$

$\frac{dv}{dt} = -\rho v - g$ separable

try on your own →
⋮

solution $\boxed{v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}}$

when $t=0$: $v(0) = v_0$

also $\lim_{t\to\infty} v(t) = \underbrace{\left(v_0 + \frac{g}{\rho}\right)\cancel{e^{-\rho t}}}_{0} - \frac{g}{\rho} = -\frac{g}{\rho} = v_\tau$

tau

v_τ is terminal velocity